Fermions and noncommutative theories

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Abstract

By using a framework where the object of noncommutativity $\theta^{\mu\nu}$ represents independent degrees of freedom, we study the symmetry properties of an extended $x + \theta$ space-time, given by the group P', which has the Poincaré group P as a subgroup. In this process we use the minimal canonical extension of the Doplicher, Fredenhagen and Roberts algebra. It is also proposed a generalized Dirac equation, where the fermionic field depends not only on the ordinary coordinates but on $\theta^{\mu\nu}$ as well. The dynamical symmetry content of such fermionic theory is discussed, and we show that its action is invariant under $\mathcal{P}$'.

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Since the seminal work of Snyder [1], space-time noncommutativity has deserved a great amount of study. Nowadays the subject is mainly associated with strings [2], noncommutative field theories(NCFT's) [3] and gravity [4], which can be related [5, 6]. In the fundamental noncommutativity relation

$$[\mathbf{x}^\mu, \mathbf{x}^\nu] = i\theta^{\mu\nu} \tag{1}$$

the object of noncommutativity $\theta^{\mu\nu}$ is usually considered as a constant matrix, what breaks Lorentz symmetry [3]. Actually, if strings have their end points on D-branes, in the presence of a constant antisymmetric tensor field background, this kind of canonical noncommutativity effectively arises. Recently it was discovered that the structure (1) is compatible with the twisted Poincaré symmetry [7], which can be an important ingredient in the construction of NCFT's. In a different perspective, $\theta^{\mu\nu}$ can also be considered as an independent operator [8]- [13], which permits to construct true Lorentz invariant theories. As too little is known about Physics at Planck scale, it seems interesting to study different structures which could arise in a very high energy regime. The last works cited above are based on a contraction of the Snyder's algebra or directly on the Doplicher, Fredenhagen and Roberts (DFR) algebra [14]. The DFR algebra assumes, besides (1), the structure

$$\begin{aligned} &[\mathbf{x}^\mu, \theta^{\alpha\beta}] = 0 \\ &[\theta^{\mu\nu}, \theta^{\alpha\beta}] = 0 \end{aligned} \tag{2}$$

although more general settings could be assumed [15]. The authors of [14] also propose the quantum conditions

$$\begin{aligned} &\theta_{\mu\nu}\theta^{\mu\nu} = 0 \\ &(\frac{1}{4} \star \theta^{\mu\nu}\theta_{\mu\nu})^2 = \lambda_P^8 \end{aligned} \tag{3}$$

where $\star\theta_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta^{\rho\sigma}$and λ_P is the Planck length. Their theory illuminates localizability in a quantum space-time, which has to be extended in order to include the object of noncommutativity as an independent set of coordinates. The same occurs in Refs. [8]- [13].

In two recent works [16, 17], the author proposes a minimal canonical extension of the DFR algebra in order to implement, in a noncommutative quantum mechanics (NCQM) framework [18], Poincaré invariance as a dynamical symmetry [19]. Of course this represents one among several possibilities of incorporating noncommutativity in quantum theories. In [17] not only the coordinates $\mathbf{x}^\mu$ and their conjugate momenta $\mathbf{p}_\mu$ are operators acting in a Hilbert space $\mathcal{H}$, but also $\theta^{\mu\nu}$ and their canonical momenta $\pi_{\mu\nu}$ are considered as Hilbert space operators as well. Besides (1), (2), the proposed extension of the DFR algebra is given by

$$
\begin{aligned}
&[\mathbf{x}^\mu, \mathbf{p}_\nu] = i\delta^\mu_\nu \\
&[\mathbf{p}_\mu, \mathbf{p}_\nu] = 0 \\
&[\theta^{\mu\nu}, \pi_{\rho\sigma}] = i\delta^{\mu\nu}_{\ \ \rho\sigma} \\
&[\pi_{\mu\nu}, \pi_{\rho\sigma}] = 0 \\
&[\mathbf{p}_\mu, \theta^{\rho\sigma}] = 0 \\
&[\mathbf{p}_\mu, \pi_{\rho\sigma}] = 0 \\
&[\mathbf{x}^\mu, \pi_{\rho\sigma}] = -\frac{i}{2}\delta^{\mu\nu}_{\ \ \rho\sigma} p_\nu
\end{aligned}
\tag{4}
$$

where $\delta^{\mu\nu}_{\ \ \rho\sigma} = \delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho$. The relations above are consistent under all possible Jacobi identities.

Now it is possible to adopt [20]

$$\mathbf{M}^{\mu\nu} = \mathbf{X}^\mu \mathbf{p}^\nu - \mathbf{X}^\nu \mathbf{p}^\mu - \theta^{\mu\sigma}\pi_\sigma^{\ \nu} + \theta^{\nu\sigma}\pi_\sigma^{\ \mu} \tag{5}$$

where [18]

$$\mathbf{X}^\mu = \mathbf{x}^\mu + \frac{1}{2}\theta^{\mu\nu}\mathbf{p}_\nu \tag{6}$$

as the generator of the Lorentz group. It closes in the appropriate algebra

$$[\mathbf{M}^{\mu\nu}, \mathbf{M}^{\rho\sigma}] = i\eta^{\mu\sigma}\mathbf{M}^{\rho\nu} - i\eta^{\nu\sigma}\mathbf{M}^{\rho\mu} - i\eta^{\mu\rho}\mathbf{M}^{\sigma\nu} + i\eta^{\nu\rho}\mathbf{M}^{\sigma\mu} \tag{7}$$

and generates the expected Lorentz transformations on the Hilbert space operators. Observe that the usual form of the ordinary Lorentz generator, given by $\mathbf{l}^{\mu\nu} = \mathbf{x}^\mu \mathbf{p}^\nu - \mathbf{x}^\nu \mathbf{p}^\mu$, fails to close in an algebra if (1) is adopted.

Now, let us define the dynamical transformation of an arbitrary operator $\mathbf{A}$ in $\mathcal{H}$ by $\delta\mathbf{A} = i[\mathbf{A}, \mathbf{G}]$. In the above expression let us choose $\mathbf{G} = \frac{1}{2}\omega_{\mu\nu}\mathbf{M}^{\mu\nu} - a^\mu \mathbf{p}_\mu + \frac{1}{2}b^{\mu\nu}\pi_{\mu\nu}$, where $\omega^{\mu\nu} = -\omega^{\nu\mu}$, a^μ and $b^{\mu\nu} = -b^{\nu\mu}$ are infinitesimal parameters. We get

$$\begin{aligned}
\delta\mathbf{x}^\mu &= \omega^\mu{}_\nu \mathbf{x}^\nu + a^\mu + \frac{1}{2}b^{\mu\nu}p_\nu \\
\delta\mathbf{X}^\mu &= \omega^\mu{}_\nu \mathbf{X}^\nu + a^\mu \\
\delta\mathbf{p}_\mu &= \omega_\mu{}^\nu \mathbf{p}_\nu \\
\delta\theta^{\mu\nu} &= \omega^\mu{}_\rho \theta^{\rho\nu} + \omega^\nu{}_\rho \theta^{\mu\rho} + b^{\mu\nu} \\
\delta\pi_{\mu\nu} &= \omega_\mu{}^\rho \pi_{\rho\nu} + \omega_\nu{}^\rho \pi_{\mu\rho} \\
\delta\mathbf{M}^{\mu\nu} &= \omega^\mu{}_\rho \mathbf{M}^{\rho\nu} + \omega^\nu{}_\rho \mathbf{M}^{\mu\rho} + a^\mu \mathbf{p}^\nu - a^\nu \mathbf{p}^\mu + b^{\mu\rho}\pi_\rho{}^\nu + b^{\nu\rho}\pi^\mu{}_\rho
\end{aligned} \tag{8}$$

which generalizes the action of the Poincaré group P in order to include θ translations. Let us refer to this group as P'. The P' transformations close in an algebra. As can be verified,

$$[\delta_2, \delta_1]\,\mathbf{y} = \delta_3\,\mathbf{y} \tag{9}$$

where the parameters composition rule is given by

$$\begin{aligned}
&\omega_3^\mu{}_\nu = \omega_1^\mu{}_\alpha \omega_2^\alpha{}_\nu - \omega_2^\mu{}_\alpha \omega_1^\alpha{}_\nu \\
&a_3^\mu = \omega_1^\mu{}_\nu a_2^\nu - \omega_2^\mu{}_\nu a_1^\nu \\
&b_3^{\mu\nu} = \omega_1^\mu{}_\rho b_2^{\rho\nu} - \omega_2^\mu{}_\rho b_1^{\rho\nu} - \omega_1^\nu{}_\rho b_2^{\rho\mu} + \omega_2^\nu{}_\rho b_1^{\rho\mu}
\end{aligned} \tag{10}$$

To understand the symmetry content displayed in (8), we observe that the Hilbert space $\mathcal{H}$ can be written as the direct product of two disjoint subspaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. The operators $\mathbf{X}^\mu$, $\mathbf{p}_\mu$ and $\mathbf{M}_1^{\mu\nu} = \mathbf{X}^\mu\mathbf{p}^\nu - \mathbf{X}^\nu\mathbf{p}^\mu$ act on $\mathcal{H}_1$, and $\theta^{\mu\nu}$, $\pi_{\mu\nu}$ and $\mathbf{M}_2^{\mu\nu} = -\theta^{\mu\sigma}\pi_\sigma{}^\nu + \theta^{\nu\sigma}\pi_\sigma{}^\mu$ act on $\mathcal{H}_2$. Both $\mathbf{M}_1^{\mu\nu}$ and $\mathbf{M}_2^{\mu\nu}$ satisfy the Lorentz algebra (5) and their symmetry properties can be read from (8). The unexpected transformation of $\mathbf{x}^\mu$ can be understood from (6), since $\mathbf{x}^\mu$ can be seen as a nonlinear combination of operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$. Now $\mathbf{p}_\mu$ and $\mathbf{M}_1^{\mu\nu}$ generate the usual Poincaré group P, which is the semidirect product of the four dimensional Lorentz grope L and the four dimensional translation group T_4. The transformation group G

acting on $\mathcal{H}_2$ can be seen as the semidirect product of the four dimensional Lorentz group and the six dimensional translation group T_6. As it is well known, P has $\mathbf{C}_1 = \mathbf{p}^2$ and $\mathbf{C}_2 = \mathbf{s}^2$ as invariant Casimir operators, where $\mathbf{s}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathbf{M}_1^{\nu\rho}\mathbf{p}^\sigma$ is the Pauli-Lubanski vector. $\mathbf{C}_3 = \pi^2$ and $\mathbf{C}_4 = \mathbf{M}_2^{\mu\nu}\pi_{\mu\nu}$ are the Casimir operators of G. A possible representation for P' can then be given by the 11×11 matrix

$$D_{P'}(\Lambda, A, B) = \begin{pmatrix} \Lambda^\mu{}_\nu & 0 & A^\mu \\ 0 & \Lambda^{[\mu}{}_\alpha \Lambda^{\nu]}{}_\beta & B^{\mu\nu} \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

acting for instance on the 11-dimensional colum vector $\begin{pmatrix} X^\mu \\ \theta^{\mu\nu} \\ 1 \end{pmatrix}$. The transformations (8) are reproduced for the infinitesimal case. In (11), $(\Lambda^{[\mu}{}_\alpha \Lambda^{\nu]}{}_\beta)$ forms the antisymmetric product (6×6) representation for L. With this structure we see that the usual classification scheme of the elementary particles accordingly to the eigenvalues of $\mathbf{C}_1$ and $\mathbf{C}_2$ is not lost. However, P and not P' can be the symmetry group when a particular theory is considered. This is the case of the theories adopting the quantum conditions (3) or the Seiberg-Witten version of noncommutative gauge theories [**?**]. In this situation P' is dynamically contracted to P.

An important point is that due to (1) the operator $\mathbf{x}^\mu$ can not be used to label possible basis in $\mathcal{H}$. However, as the components of $\mathbf{X}^\mu$ commute, as can be verified from (6), their eigenvalues can be used for such purpose. To simplify the notation, let us denote by x and θ the eigenvalues of $\mathbf{X}$ and θ in what follows. In [17] we have considered these points with some detail and have proposed a way for constructing some actions representing possible field theories in this extended $x+\theta$ space-time. One of such actions has been given by

$$S = -\int d^4\, x\, d^6\theta\, \Omega(\theta)\, \frac{1}{2}\Big\{ \partial^\mu\phi\partial_\mu\phi + \frac{\lambda^2}{4}\,\partial^{\mu\nu}\phi\partial_{\mu\nu}\phi + m^2\,\phi^2 \Big\} \tag{12}$$

where λ is a parameter with dimension of length, as the Planck length, which has to be introduced by dimensional reasons and $\Omega(\theta)$ is a scalar weight function used in [8]- [13] in order to make the connection between the $D = 4 + 6$ and the $D = 4$ formalisms. Also $\Box = \partial^\mu\partial_\mu$, with $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\Box_\theta = \frac{1}{2}\partial^{\mu\nu}\partial_{\mu\nu}$, where $\partial_{\mu\nu} = \frac{\partial}{\partial\theta^{\mu\nu}}$. $\eta^{\mu\nu} = diag(-1, 1, 1, 1)$.

The corresponding Lagrange equation reads

$$\begin{aligned} \frac{\delta S}{\delta \phi} &= \Omega\,(\Box - m^2)\phi + \frac{\lambda^2}{2}\partial_{\mu\nu}(\Omega\,\partial^{\mu\nu}\phi) \\ &= 0 \end{aligned} \tag{13}$$

and the action (12) is invariant under the transformation

$$\delta\phi = -(a^\mu + \omega^\mu{}_\nu x^\nu)\,\partial_\mu\phi - \frac{1}{2}(b^{\mu\nu} + 2\omega^\mu{}_\rho\theta^{\rho\nu})\,\partial_{\mu\nu}\phi \tag{14}$$

when Ω is considered as a constant. If Ω is a non constant scalar function of θ, the above transformation is only a symmetry of (12) when $b^{\mu\nu}$ vanishes, what dynamically contracts P' to P [17]. We observe that (14) closes in an algebra, as in (9), with the same composition rule defined in (10). That equation defines how a scalar field transforms in the $x + \theta$ space under the action of $\mathcal{P}'$.

In what follows we are going to show how to introduce fermions in this $x + \theta$ extended space. To reach this goal, let us first observe that $\mathcal{P}$' is a subgroup of the Poincaré group $\mathcal{P}_{10}$ in $D = 10$. Denoting the indices $A, B, ...$ as space-time indices in $D = 10$, $A, B, .. = 0, 1, ..., 9$, a vector Y^A would transform under $\mathcal{P}_{10}$ as $\delta Y^A = \omega^A{}_B Y^B + \Delta^A$, where the 45 ω's and 10 Δ's are infinitesimal parameters. If one identifies the last six $A, B, ..$ indices with the macro-indices $\mu\nu$, $\mu, \nu, .. = 0, 1, 2, 3$, considered as antisymmetric quantities, the transformation relations given above are rewritten as

$$\begin{aligned} \delta Y^\mu &= \omega^\mu{}_\nu Y^\nu + \frac{1}{2}\omega^\mu{}_{\alpha\beta} Y^{\alpha\beta} + \Delta^\mu \\ \delta Y^{\mu\nu} &= \omega^{\mu\nu}{}_\alpha Y^\alpha + \frac{1}{2}\omega^{\mu\nu}{}_{\alpha\beta} Y^{\alpha\beta} + \Delta^{\mu\nu} \end{aligned} \tag{15}$$

With this notation, the (diagonal) $D = 10$ Minkowski metric is rewritten as $\eta^{AB} = (\eta^{\mu\nu}, \eta^{\alpha\beta,\gamma\delta})$ and the ordinary Clifford algebra $\{\Gamma^A, \Gamma^B\} = -2\eta^{AB}$ as

$$\begin{aligned} \{\Gamma^\mu, \Gamma^{\alpha\beta}\} &= 0 \\ \{\Gamma^\mu, \Gamma^\nu\} &= -2\eta^{\mu\nu} \\ \{\Gamma^{\mu\nu}, \Gamma^{\alpha\beta}\} &= -2\eta^{\mu\nu,\alpha\beta} \end{aligned} \tag{16}$$

This is just a cumbersome way of writing usual $D = 10$ relations [2]. Now, by identifying Y^A with $(x^\mu, \frac{1}{\lambda}\theta^{\alpha\beta})$, where λ is some parameter with length dimension, we see from the structure given above that the allowed transformations in $\mathcal{P}$' are those of $\mathcal{P}_{\,10}$, submitted to the conditions

$$\begin{aligned}
\omega^{\mu\nu}{}_{\alpha} &= \omega_{\mu\nu}{}^{\alpha} = 0 \\
\omega^{\mu\nu}{}_{\alpha\beta} &= 4\,\omega^{[\mu}{}_{\alpha}\delta^{\nu]}{}_{\beta} \\
\Delta^\mu &= a^\mu \\
\Delta^{\alpha\beta} &= \frac{1}{\lambda}b^{\alpha\beta}
\end{aligned} \tag{17}$$

obviously keeping the identification between ω^{AB} and $\omega^{\mu\nu}$ when $A = \mu$ and $B = \nu$. Of course we have now only 6 independent ω's and 10 a's and b's. With the relations given above it is possible to extract the "square root" of the generalized Klein-Gordon equation (13)

$$(\Box + \lambda^2\Box_\theta - m^2)\phi = 0 \tag{18}$$

assuming here that Ω is a constant. This gives just the generalized Dirac equation

$$\Big[i(\Gamma^\mu\partial_\mu + \frac{\lambda}{2}\Gamma^{\alpha\beta}\partial_{\alpha\beta}) - m\Big]\psi = 0 \tag{19}$$

Actually, by applying from the left by the operator $\Big[i(\Gamma^\nu\partial_\mu + \frac{\lambda}{2}\Gamma^{\alpha\beta}\partial_{\alpha\beta}) + m\Big]$ on (19), after using (16) we observe that ψ satisfies the generalized Klein-Gordon equation (18) as well. The covariance of the generalized Dirac equation (19) can also be proved. Fist we note that the operator

$$M^{\mu\nu} = \frac{i}{4}\Big([\Gamma^\mu, \Gamma^\nu] + [\Gamma^{\mu\alpha}, \Gamma^\nu{}_\alpha]\Big) \tag{20}$$

gives the desired representation for the $SO(1,3)$ generators, because it not only closes in the Lorentz algebra (7), but also satisfies the commutation relations

$$[\Gamma^\mu, M_{\alpha\beta}] = 2i\delta^\mu_{[\alpha}\Gamma_{\beta]}$$
$$[\Gamma^{\mu\nu}, M_{\alpha\beta}] = 2i\delta^\mu_{[\alpha}\Gamma_{\beta]}^{\ \nu} - 2i\delta^\nu_{[\alpha}\Gamma_{\beta]}^{\ \mu} \tag{21}$$

With these relations it is possible to prove that (19) is indeed covariant under the Lorentz transformations given by

$$\psi(x', \theta') = exp(-\frac{i}{2}\Lambda^{\mu\nu}M_{\mu\nu})\psi(x, \theta) \tag{22}$$

By considering the complete $\mathcal{P}$' group, we observe that the infinitesimal transformations of ψ are given by

$$\delta\psi = -\Big[(a^\mu + \omega^\mu{}_\nu x^\nu)\partial_\mu + \frac{1}{2}(b^{\mu\nu} + 2\omega^\mu{}_\rho\theta^{\nu\rho})\partial_{\mu\nu} + \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\Big]\psi \tag{23}$$

which closes in the $\mathcal{P}$' algebra with the same composition rule given by (10), what can be shown after a little algebra. At last we can show that also here there are conserved Noether's currents associated with the transformation (23), once we observe that the equation (19) can be derived from the action

$$S = \int d^4 x\, d^6\theta\, \Omega(\theta)\, \bar{\psi}\Big[i(\Gamma^\mu\partial_\mu + \frac{\lambda}{2}\Gamma^{\alpha\beta}\partial_{\alpha\beta}) - m\Big]\psi \tag{24}$$

where we are considering $\Omega = \theta_0^{-6}$ and $\bar{\psi} = \psi^\dagger\Gamma^0$. First we note that (suppressing trivial θ_0^{-6} trivial factors)

$$\begin{aligned}
\frac{\delta^L S}{\delta\bar{\psi}} &= \Big[i(\Gamma^\mu\partial_\mu + \frac{\lambda}{2}\Gamma^{\alpha\beta}\partial_{\alpha\beta}) - m\Big]\psi \\
\frac{\delta^R S}{\delta\psi} &= -\bar{\psi}\Big[i(\Gamma^\mu\overleftarrow{\partial}_\mu + \frac{\lambda}{2}\Gamma^{\alpha\beta}\overleftarrow{\partial}_{\alpha\beta}) + m\Big]
\end{aligned} \tag{25}$$

where $L(R)$ derivatives act from the left(right). The current $(j^\mu, j^{\mu\nu})$, as in [17], is here written as

$$j^\mu = \frac{\partial^R\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi + \delta\bar{\psi}\frac{\partial^L\mathcal{L}}{\partial\partial_\mu\bar{\psi}} + \mathcal{L}\delta x^\mu$$
$$j^{\mu\nu} = \frac{\partial^R\mathcal{L}}{\partial\partial_{\mu\nu}\psi}\delta\psi + \delta\bar{\psi}\frac{\partial^L\mathcal{L}}{\partial\partial_{\mu\nu}\bar{\psi}} + \mathcal{L}\delta\theta^{\mu\nu} \tag{26}$$

where

$$\delta\bar{\psi} = -\bar{\psi}\Big[\overleftarrow{\partial}_\mu (a^\mu + \omega^\mu{}_\nu x^\nu) + \overleftarrow{\partial}_{\mu\nu}\frac{1}{2}(b^{\mu\nu} + 2\omega^\mu{}_\rho \theta^{\nu\rho}) - \frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}\Big] \tag{27}$$

$\delta\psi$ is given by (23) and δx^μ and $\delta\theta^{\mu\nu}$ have the same form found in (8). After a long but direct calculation one can show that

$$\partial_\mu j^\mu + \partial_{\mu\nu} j^{\mu\nu} = -\Big(\delta\bar{\psi}\frac{\delta^L S}{\delta\bar{\psi}} + \frac{\delta^R S}{\delta\psi}\delta\psi\Big) \tag{28}$$

which vanishes on shell, proving the invariance of the action (24) under $\mathcal{P}$'. By the reasons pointed through this work, it could be dynamically contracted to P, preserving the usual Casimir invariant structure characteristic of ordinary quantum field theories.

Due to (28) there is a conserved charge

$$Q = \int d^3x d^6\theta \, j^0 \tag{29}$$

for each one of the specific transformations encoded in (26). Actually, $\dot{Q} = -\int d^3x d^6\theta \, (\partial_i \, j^i + \partial_{\mu\nu} j^{\mu\nu})$ vanishes as a consequence of the divergence theorem. By considering only x^μ translations, we can write $j^0 = j^0_\mu a^\mu$, permitting to define the momentum operator $P_\mu = -\int d^3x d^6\theta j^0_\mu$. Also by considering $\theta^{\mu\nu}$ translations and Lorentz transformations, we can derive in a similar way an explicit form for the other generators of $\mathcal{P}$', here denoted by $\Pi_{\mu\nu}$ and $J_{\mu\nu}$. Under an appropriate bracket structure, as teaches us the Noether's theorem, these conserved charges will generate the transformations (23) and (27).

We close this work by observing that we have been able to introduce fermions satisfying a generalized Dirac equation, which is covariant under the action of the extended Poincaré group P'. That equation has been derived through a variational principle whose action is dynamically invariant under P'. This can clarify possible rules played by theories involving noncommutativity in a way compatible with Relativity. Of course this is just a little step toward a field theory quantization program in this extended $x + \theta$ space-time. This last point is under study and possible results will be reported elsewhere.

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